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Bifractional inequalities and convex cones

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Abstract

In this paper we give a systematic study of a class of linear inequalities related to convex cones in linear spaces. In particular, Chebyshev and Andersson type inequalities are discussed. Some classical and new inequalities are derived from the results.

Keywords: Convex cone; Dual cone; Bifractional inequality; Chebyshev inequality; Andersson inequality; Inequality of G. Seitz

1. Introduction and summary

Let V be a real linear space equipped with an inner product $\langle \cdot, \cdot \rangle$. In this paper we study the inequalities of the form

$$\langle z, x\rangle \langle y, v\rangle \leqslant \langle z, y\rangle \langle x, v\rangle, \tag{1}$$

where v, x, y, z are certain vectors in V. If $\langle v, x\rangle\langle v, y\rangle > 0$ then an equivalent form of (1) is

$$\frac{\langle z, x\rangle}{\langle v, x\rangle} \leqslant \frac{\langle z, y\rangle}{\langle v, y\rangle} \tag{2}$$

(with the reverse inequality if $\langle v, x\rangle\langle v, y\rangle < 0$). This is a *bifractional inequality*.

For instance, if V is $\mathbb{R}^n$ with the standard inner product and if y and z are nonincreasing (or nondecreasing) sequences with $x = v = (1, \ldots, 1) \in \mathbb{R}^n$, then (1) becomes *Chebyshev sum inequality* (see [9,13,14]). Likewise, if $x = (1, 2, \ldots, n)$ then (1) leads to *Andersson type inequalities* (see [1,2,5]). A similar problem for convex sequences in $\mathbb{R}^n$ has been studied by Mercer [4] and Gavrea [3]. See also [7] for the case of convex sequences of order r.

The aim of this paper is to provide conditions on the vectors v, x, y and z under which (1)–(2) are satisfied. Utilizing the linearity of (1) in z and y, we give a framework for our problem based on the duality of convex cones. Our approach extends recent ideas due to Fink [1] and Mercer [5].

The paper is organized as follows. In Section 2 we provide basic notions related to cones. The results are collected in Section 3. We begin with a cone method for solving (1) (see Theorem 3.1). Subsequently, we consider the case of polyhedral cones related to bases of the space. In this context we introduce the notion of a separable vector on two given sets of indices. This concept extends the notion of synchronicity of vectors (see Remark 3.8). Theorem 3.5 is a completion of Theorem 3.1 for separable vectors. Here we develop ideas of Rychlik [9] and Toader [10]. More

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particular classes of vectors are treated at the end of Section 3 (see Corollary 3.7). Examples illustrating the theory are also given.

Section 4 deals with applications. Here we discuss Chebyshev and Andersson type inequalities. In particular, we derive a result of Toader [13, Theorem 1] for star shaped sequences. This is closely related to a discrete version of a recent result of Fink [1, Theorem 2]. The discussion of the problem for nonorthogonal bases and concave sequences is given (see Example 4.5).

Further applications are collected in Section 5. We analyze some results of Toader [11,12] and of Wang and Luo [15] on inequalities of Seitz and of Fujiwara. In our approach, we replace the property of synchronicity by the separability of pairs of vectors. This allows to extend the range of applicability of the above-mentioned inequalities.

2. Preliminaries

Throughout this paper, V is a real linear space with inner product $\langle \cdot, \cdot \rangle$. A *convex cone* is a nonempty set $C \subset V$ such that $\alpha C + \beta C \subset C$ for all nonnegative scalars α and β. For a (nonempty) subset G of V, the symbol cone G stands for the convex cone of all nonnegative finite combinations of vectors in G.

The *dual cone* of C is the cone defined by

$$\operatorname{dual} C = \{w \in V : \langle u, w \rangle \geqslant 0 \text{ for all } u \in C\}.$$

It is known [8, p. 121] that

$$\operatorname{dual}\operatorname{dual} C = C \tag{3}$$

for any closed convex cone $C \subset V$ (if $\dim V < \infty$).

Let $C \subset V$ be a convex cone. For given $x, y \in V$ we write $y \preccurlyeq_C x$ if $x - y \in C$. The relation $\preccurlyeq_C$ is a (cone) preordering on V. Notice that if (1) holds for every z in a convex cone C then one has

$$\langle y, v \rangle x \preccurlyeq_D \langle x, v \rangle y,$$

where $D = \operatorname{dual} C$.

3. Results

We start with a general method for finding vectors satisfying (1).

Theorem 3.1. *Let x and v be given vectors in V. For vectors $y, z \in V$, the following statements are mutually equivalent.*

(i) *The inequality*

$$\langle z, x \rangle \langle y, v \rangle \leqslant \langle z, y \rangle \langle x, v \rangle \tag{4}$$

holds.

(ii) *There exists a convex cone $C \subset V$ such that $z \in C$ and*

$$\langle x, v \rangle y - \langle y, v \rangle x \in \operatorname{dual} C. \tag{5}$$

(iii) *There exists a convex cone $C \subset V$ such that*

$$(z, y) \in C \times \operatorname{dual} L_{x,v} C, \tag{6}$$

where $\times$ denotes the Cartesian product and $L_{x,v}(\cdot) = \langle x, v \rangle (\cdot) - \langle \cdot, x \rangle v$.

(iv) *There exists a convex cone* $C \subset V$ *such that*

$$(z, y) \in (C + \operatorname{span} v) \times \operatorname{dual} L_{x,v}C. \tag{7}$$

Proof. (i) $\Rightarrow$ (ii). Set $C = \operatorname{cone} z$. Clearly $z \in C$. Since the identity

$$\langle z, y\rangle\langle x, v\rangle - \langle z, x\rangle\langle y, v\rangle = \langle z, \langle x, v\rangle y - \langle y, v\rangle x\rangle \tag{8}$$

holds, it follows from (4) that (5) is valid.

(ii) $\Rightarrow$ (iii). By (5), $L_{v,x}y \in \operatorname{dual} C$, where $L_{v,x}(\cdot) = \langle v, x\rangle(\cdot) - \langle \cdot, v\rangle x$. Hence $\langle C, L_{v,x}y\rangle \geqslant 0$, or, equivalently, $\langle L^{\mathrm{T}}_{v,x}C, y\rangle \geqslant 0$, which means that

$$y \in \operatorname{dual} L^{\mathrm{T}}_{v,x}C = \operatorname{dual} L_{x,v}C,$$

because $L^{\mathrm{T}}_{v,x} = L_{x,v}$, where $(\cdot)^{\mathrm{T}}$ denotes the transpose. Therefore (ii) implies (iii).

(iii) $\Rightarrow$ (iv). This implication is obvious, since $C \subset C + \operatorname{span} v$.

(iv) $\Rightarrow$ (i). We have $z \in C + \operatorname{span} v$ and $y \in \operatorname{dual} L_{x,v}C$. There exist $c \in C$ and $\mu \in \mathbb{R}$ such that $z = c + \mu v$. Moreover, $L_{v,x}y \in \operatorname{dual} C$. In consequence, $\langle z - \mu v, L_{v,x}y\rangle \geqslant 0$. Simultaneously, $\langle v, L_{v,x}y\rangle = 0$, so $\langle z, L_{v,x}y\rangle \geqslant 0$, which gives (i). This completes the proof of Theorem 3.1. $\square$

In conclusion, Theorem 3.1 asserts that for any given vectors x and v in V, inequality (4) is satisfied if and only if

$$(z, y) \in \bigcup_C C \times \operatorname{dual} L_{x,v}C, \tag{9}$$

where C runs over the class of *all* convex cones in V. In particular, if $\{C_j : j \in J\}$ is a class of convex cones in V then the condition

$$z \in \bigcup_{j\in J} C_j \quad \text{and} \quad y \in \bigcap_{j\in J} \operatorname{dual} L_{x,v}C_j$$

guarantees that (9) and (4) are met.

Example 3.2. An inequality of S. Haber (see [4, p. 1]) asserts that

$$\sum_{k=0}^{n} \left(\frac{1}{n+1} - \frac{1}{2^n}\binom{n}{k}\right) a^k \geqslant 0 \quad \text{for } 0 \leqslant a \in \mathbb{R}, \tag{10}$$

where $\binom{n}{k} = n!/k!(n-k)!$ is the Newton symbol.

To express the above result in the terminology of Theorem 3.1, we let V denote $\mathbb{R}^{n+1}$, and C denote the cone of all convex sequences in $\mathbb{R}^{n+1}$. Also, we set $z = (a^0, a^1, \ldots, a^n)$, $x = \left(\binom{n}{0}, \binom{n}{1}, \ldots, \binom{n}{n}\right)$ and $y = v = (1, 1, \ldots, 1) \in \mathbb{R}^{n+1}$.

It is clear that $\langle y, v\rangle = n + 1$ and $\langle v, x\rangle = 2^n$. Since $z \in C$, condition (5) takes the equivalent form

$$\frac{L_{v,x}y}{\langle y, v\rangle\langle v, x\rangle} = \frac{y}{n+1} - \frac{x}{2^n} \in \operatorname{dual} C.$$

In other words, (10) reduces to (2), or, equivalently, to (4).

Remark that in the case $x = v\ (\neq 0)$ we get

$$L_{x,v} = \|v\|^2\left(\mathrm{id} - \frac{\langle \cdot, v\rangle v}{\|v\|^2}\right) = \|v\|^2(\mathrm{id} - P_v),$$

where id is the identity map and P_v is the orthoprojector onto the subspace spanned by the vector v.

Example 3.3. Take $V = \mathbb{R}^n$ and $x = v = (1, \ldots, 1) \in \mathbb{R}^n$. Then, in matrix notation, $L_{x,v} = n(\mathrm{id} - (1/n)E)$, where id is the n-by-n identity matrix and E is the n-by-n matrix of all ones. Here the operator $(1/n)E$ is the orthoprojector onto the subspace spanned by v and $\mathrm{id} - (1/n)E$ is the orthoprojector onto the subspace $V_0 = \{u \in \mathbb{R}^n : \sum u_i = 0\}$.

Let C be the cone of *nonincreasing* sequences, that is

$$C=\{c=(c_1,\ldots,c_n)\in\mathbb{R}^n : c_1\geqslant\cdots\geqslant c_n\}.$$

It is not hard to check that $L_{x,v}C=\left\{c\in C:\sum c_i=0\right\}$. Hence dual $L_{x,v}C=\operatorname{dual}C+\operatorname{span}v$. Notice that $C\subset\operatorname{dual}C+\operatorname{span}v$. In consequence, $z,y\in C$ implies (9), which gives (4), i.e.,

$$\sum_{k=1}^n z_k\sum_{k=1}^n y_k\leqslant n\sum_{k=1}^n z_k y_k. \tag{11}$$

This is *Chebyshev sum inequality* (see [13]). See Example 3.6 and Section 4 for a different approach to this inequality with other assumptions on vectors z and y.

According to Theorem 3.1, part (ii), it is natural to employ the following general method for proving the fundamental inequality (4).

Fix any $x,v\in V$. Suppose that $\{D_j : j\in J\}$ is a class of convex cones in V such that

$$V=\bigcup_{j\in J}D_j. \tag{12}$$

Let C_j stand for the dual cone of D_j. Take any $y\in V$. Then there exists an index $j_0\in J$ such that

$$\langle x,v\rangle y-\langle y,v\rangle x\in D_{j_0}.$$

Choose any $z\in C_{j_0}$. By (3), $D_{j_0}=\operatorname{dual}C_{j_0}$ (if $\dim V<\infty$). So, condition (ii) of Theorem 3.1 is fulfilled. Consequently, inequality (4) is true.

We will explore the above idea in the sequel.

From now on, we will consider the problem of solving (4) in the context of polyhedral cones. For the statement of our results we need some notation.

Assume V is a finite-dimensional inner product space. Let $e=\{e_1,\ldots,e_n\}$ be a basis in V, and let $d=\{d_1,\ldots,d_n\}$ be the dual basis of e, that is $\langle e_i,d_j\rangle$ equals one (zero) if $i=j$ (resp. $i\neq j$). Then for $u,w\in V$ we have

$$u=\sum_{i=1}^n\langle e_i,u\rangle d_i\quad\text{and}\quad w=\sum_{i=1}^n\langle d_i,w\rangle e_i. \tag{13}$$

In consequence, one obtains

$$\langle u,w\rangle=\sum_{i=1}^n\langle e_i,u\rangle\langle d_i,w\rangle. \tag{14}$$

We say that a vector $u\in V$ is *e-positive* (*e-negative*), if $\langle e_i,u\rangle>0$ (resp. $\langle e_i,u\rangle<0$) for all $1\leqslant i\leqslant n$. It follows from (13) that the e-positivity of u implies $u\in\operatorname{cone}\{d_1,\ldots,d_n\}$.

Denote $I=\{1,\ldots,n\}$. Let I_1 and I_2 be two sets of indices such that $I_1\cup I_2=I$. (It is possible that I_1 or I_2 is empty and that $I_1\cap I_2$ is nonempty.) We define

$$C_e(I_1,I_2)=\operatorname{dual}\operatorname{cone}\{e_i : i\in I_1\}\cup\{-e_j : j\in I_2\}. \tag{15}$$

By (13) we obtain

$$C_e(I_1,I_2)=\operatorname{cone}\{d_i : i\in I_1\}\cup\{-d_j : j\in I_2\}. \tag{16}$$

Observe that

$$\operatorname{dual}C_e(I_1,I_2)=C_d(I_1,I_2). \tag{17}$$

Since d is a basis in V,

$$V = \bigcup_{I_1 \cup I_2 = I} C_e(I_1, I_2)$$

(cf. (12)). In fact, for any $u \in V$, we have $u \in C_e(I_1, I_2)$ for

$$I_1 = I_1(u) = \{i \in I : \langle e_i, u\rangle \geqslant 0\} \quad \text{and} \quad I_2 = I_2(u) = \{j \in I : \langle e_j, u\rangle \leqslant 0\}.$$

Consider any vector $v \in V$ and scalar μ. A vector $z \in V$ is said to be μ, v-*separable* on I_1 and I_2 (with respect to the basis e), if

$$\langle e_i, z - \mu v\rangle \geqslant 0 \quad \text{for } i \in I_1 \quad \text{and} \quad \langle e_j, z - \mu v\rangle \leqslant 0 \quad \text{for } j \in I_2. \tag{18}$$

Equivalently, (18) states that

$$z - \mu v \in C_e(I_1, I_2). \tag{19}$$

In other words, z is μ, v-separable on I_1 and I_2 w.r.t. e if and only if

$$\max_{j \in I_2} \frac{\langle e_j, z\rangle}{\langle e_j, v\rangle} \leqslant \mu \leqslant \min_{i \in I_1} \frac{\langle e_i, z\rangle}{\langle e_i, v\rangle} \tag{20}$$

whenever v is e-positive (with the reverse inequalities and min and max interchanged if v is e-negative).

A vector $z \in V$ is said to be v-*separable* on I_1 and I_2 (w.r.t. e), if z is μ, v-separable on I_1 and I_2 for some μ. By (20), z is v-separable on I_1 and I_2 w.r.t. e if and only if

$$\max_{j \in I_2} \frac{\langle e_j, z\rangle}{\langle e_j, v\rangle} \leqslant \min_{i \in I_1} \frac{\langle e_i, z\rangle}{\langle e_i, v\rangle} \tag{21}$$

provided v is e-positive.

We denote

$$S_e(v; I_1, I_2) = \{z \in V : z \text{ is } v\text{-separable on } I_1 \text{ and } I_2 \text{ w.r.t. } e\}.$$

In light of (19),

$$S_e(v; I_1, I_2) = C_e(I_1, I_2) + \operatorname{span} v. \tag{22}$$

Therefore $S_e(v; I_1, I_2)$ is a convex cone.

Example 3.4. Let $V = \mathbb{R}^n$ and let $z = (z_1, \ldots, z_n)$ be a nonincreasing sequence. Assume $v = (1, \ldots, 1) \in \mathbb{R}^n$. Then z is v-separable (w.r.t. the standard orthonormal basis in $\mathbb{R}^n$) on the index sets $I_1 = \{1, \ldots, p\}$ and $I_2 = \{p, p+1, \ldots, n\}$ for each $1 \leqslant p \leqslant n$.

If $\langle x, v\rangle = 0$ then (4) is trivial. Therefore without lost of generality we can assume that $\langle x, v\rangle \neq 0$.

We are now in a position to give Theorem 3.5. Its idea is based on some recent results due to Fink [1] and Mercer [5].

Theorem 3.5. *Let x and v be given vectors in V with $\langle x, v\rangle \neq 0$. Let $e = \{e_1, \ldots, e_n\}$ be a basis in V, and let $d = \{d_1, \ldots, d_n\}$ be the dual basis of e. Suppose that I_1 and I_2 are arbitrary index sets with $I_1 \cup I_2 = I$, where $I = \{1, 2, \ldots, n\}$.*

Assume $y \in V$ and denote $\lambda = \langle y, v\rangle / \langle x, v\rangle$. The following two statements are equivalent:

(i) *The vector y is λ, x-separable on I_1 and I_2 w.r.t. d if $\langle x, v\rangle > 0$ (or on I_2 and I_1 w.r.t. d if $\langle x, v\rangle < 0$).*
(ii) *Inequality* (4) *holds for all $z \in S_e(v; I_1, I_2)$.*

Proof. We consider the case $\langle x, v\rangle > 0$ only, because the alternative one is similar. Denote $w = \langle x, v\rangle y - \langle y, v\rangle x$.

(i) $\Rightarrow$ (ii). Fix any $z \in S_e(v; I_1, I_2)$. By (22), there exist $u \in C_e(I_1, I_2)$ and $\mu \in \mathbb{R}$ such that $z = u + \mu v$. On the other hand, by (i) and (19), $y - \lambda x \in C_d(I_1, I_2)$. Hence

$$w = \langle x, v\rangle (y - \lambda x) \in C_d(I_1, I_2).$$

Utilizing (17), we get $\langle u, w\rangle \geqslant 0$. Moreover, it is easy to verify that $\langle \mu v, w\rangle = 0$. In consequence, $\langle z, w\rangle \geqslant 0$, which gives (4).

(ii) $\Rightarrow$ (i). Using (ii), we derive

$$\langle z, w\rangle \geqslant 0 \quad \text{for all } z \in C_e(I_1, I_1),$$

because $C_e(I_1, I_1) \subset S_e(v; I_1, I_1)$ by (22). Hence

$$w \in \operatorname{dual} C_e(I_1, I_2) = C_d(I_1, I_2),$$

the last equality by (17). Therefore

$$y - \lambda x = \frac{1}{\langle x, v\rangle} w \in C_d(I_1, I_2).$$

Employing (19), we see that y is λ, x-separable on I_1 and I_2 w.r.t. d, as required. $\square$

Example 3.6. As in Example 3.3, consider $V = \mathbb{R}^n$ and $x = v = (1, \ldots, 1) \in \mathbb{R}^n$. Let e be the basis in $\mathbb{R}^n$ consisting of the vectors

$$e_i = (\underbrace{1, \ldots, 1}_{i \text{ times}}, 0, \ldots, 0), \quad i = 1, \ldots, n.$$

The dual basis d of e has the form

$$d_i = (\underbrace{0, \ldots, 0}_{i-1 \text{ times}}, 1, -1, 0, \ldots, 0), \quad i = 1, \ldots, n-1 \quad \text{and} \quad d_n = (0, \ldots, 0, 1).$$

In this situation the condition $z \in S_e(v; I_1, I_2)$ means that

$$\frac{\sum_{k=1}^{j} z_k}{j} \leqslant \frac{\sum_{k=1}^{i} z_k}{i} \quad \text{for } i \in I_1 \text{ and } j \in I_2 \tag{23}$$

(see (21)). On the other hand, the requirement that y is λ, x-separable on I_1 and I_2 w.r.t. d is equivalent to the condition (cf. (18))

$$y_j \leqslant y_{j+1} \quad \text{and} \quad y_i \geqslant y_{i+1} \ \text{ for } i \in I_1 \text{ and } j \in I_2 \tag{24}$$

with the convention that $y_{n+1} = \lambda = (1/n)\sum_{k=1}^{n} y_k$. It now follows from Theorem 3.5 that (23)–(24) imply Chebyshev inequality (11). The case $I_1 = \{1, \ldots, n-1\}$ and $I_2 = \{n\}$ leads to a result of Rychlik [9] (cf. [13, Theorem B]). Namely, (23) and (24) become

$$\frac{\sum_{k=1}^{n} z_k}{n} \leqslant \frac{\sum_{k=1}^{i} z_k}{i} \quad \text{for } 1 \leqslant i \leqslant n-1$$

and

$$y_i \geqslant y_{i+1} \quad \text{for } 1 \leqslant i \leqslant n-1.$$

In particular, if y is nonincreasing and z is *nonincreasing in mean*, i.e., the sequence $\left\{ \left(\sum_{k=1}^{i} z_k \right) / i \right\}_{i=1}^{n}$ is nonincreasing, then (11) is valid.

In the previous example the case $\langle d_i, x\rangle = 0$ has appeared. We now apply Theorem 3.5 in the situation when the vectors v and x are positive or negative. Then the property of separability can be restated in a fractional form according to (20) and (21).

Corollary 3.7. *Under the hypothesis of Theorem* 3.5, *let v be e-positive or e-negative and let x be d-positive or d-negative such that $\langle x, v\rangle > 0$. Assume there exist index sets I_1 and I_2 with $I_1 \cup I_2 = I$ such that*

$$\frac{\langle e_j, z\rangle}{\langle e_j, v\rangle} \leqslant \frac{\langle e_i, z\rangle}{\langle e_i, v\rangle} \quad \text{for } i \in I_1 \text{ and } j \in I_2 \tag{25}$$

and

$$\frac{\langle d_j, y\rangle}{\langle d_j, x\rangle} \leqslant \lambda \leqslant \frac{\langle d_i, y\rangle}{\langle d_i, x\rangle} \quad \text{for } i \in I_1 \text{ and } j \in I_2 \tag{26}$$

provided the denominators of the fractions are positive (*or the reverse inequalities hold if the denominators are negative*), *where $\lambda = \langle y, v\rangle / \langle x, v\rangle$.*

Then inequality (4) *is valid for v, x, y and z.*

It is worth emphasing that for given $y \in V$ there exist

$$I_1(y) = \left\{ i \in I : \lambda \leqslant \frac{\langle d_i, y\rangle}{\langle d_i, x\rangle} \right\} \quad \text{and} \quad I_2(y) = \left\{ j \in I : \frac{\langle d_j, y\rangle}{\langle d_j, x\rangle} \leqslant \lambda \right\} \tag{27}$$

such that (26) holds (provided $\langle d_i, x\rangle > 0$ for all $i \in I$; the alternative case is similar). So, in order to get (4), it is sufficient to choose any $z \in V$ so that (25) is satisfied for I_1 and I_2 defined by (27). If it is known that the fractions of (25) and of (26) form *monotone* sequences of the same type, then the knowledge of I_1 and I_2 is superfluous. More general, if the sequences of the fractions are *similarly ordered* (cf. [2]) (or, in other terminology, the pairs (z, v) and (y, x) are *synchronous* (cf. [11])), that is

$$\left(\frac{\langle e_i, z\rangle}{\langle e_i, v\rangle} - \frac{\langle e_j, z\rangle}{\langle e_j, v\rangle} \right) \left(\frac{\langle d_i, y\rangle}{\langle d_i, x\rangle} - \frac{\langle d_j, y\rangle}{\langle d_j, x\rangle} \right) \geqslant 0 \quad \text{for all } i, j \in I, \tag{28}$$

(for positive both v and x), then (25)–(26) are satisfied for certain index sets I_1 and I_2. For example, if $y = z$, $x = v$ and $d_i = e_i$, then (28) holds automatically, and, in consequence, (4) takes the form of the Cauchy–Schwarz inequality. See the next sections for further examples illustrating Corollary 3.7.

An interesting case is when $v = d_{i_0}$ for some $i_0 \in I$. For instance, if the functions

$$I \ni i \to \frac{\langle e_i, z\rangle}{\langle e_i, v\rangle} \in \mathbb{R} \quad \text{and} \quad I \ni i \to \frac{\langle d_i, y\rangle}{\langle d_i, x\rangle} \in \mathbb{R}$$

(with positive denominators) take their maximal (minimal) values at i_0, then (25)–(26) are fulfilled for $I_1 = \{i_0\}$ and $I_2 = I \backslash \{i_0\}$ (resp. for $I_2 = \{i_0\}$ and $I_1 = I \backslash \{i_0\}$) (cf. [13, Theorem 3] where the functions take their maximal values at $i_0 = n$).

Remark 3.8. The synchronicity (28) is a stronger condition then the property of separability (25) and (26). That is, (28) implies (25) and (26) for the index sets I_1 and I_2 defined by (27), but not vice versa.

Example 3.9. For even n take $V = \mathbb{R}^n$ with the standard orthonormal basis $e = d$. Set $x = v = (1, \ldots, 1) \in \mathbb{R}^n$ and $y = (y_1, \ldots, y_n)$ with $y_k = k$ for $k = 1, \ldots, n$, and $z = (z_1, \ldots, z_n)$ with

$$z_k = \begin{cases} \dfrac{n}{2} + 1 - k & \text{for } k = 1, \ldots, \dfrac{n}{2}, \\ \dfrac{3n}{2} + 1 - k & \text{for } k = \dfrac{n}{2} + 1, \ldots, n. \end{cases}$$

Evidently,

$$\lambda = \frac{\langle y, v\rangle}{\langle x, v\rangle} = \frac{1}{n}\sum_{k=1}^{n} y_k = \frac{n}{2} + \frac{1}{2},$$

and conditions (25)–(26) are fulfilled for $I_1 = \{n/2+1, \ldots, n\}$ and $I_2 = \{1, \ldots, n/2\}$. However, (28) does not hold, because $(z_1 - z_{n/2})(y_1 - y_{n/2}) < 0$.

4. Applications for Chebyshev and Andersson–Toader's inequalities

In this section we shall present some examples and applications. We consider the special case when V is the n-dimensional Euclidean space $\mathbb{R}^n$ equipped with the standard inner product.

Let e be the standard basis in $\mathbb{R}^n$ consisting of the vectors

$$e_i = (\underbrace{0, \ldots, 0}_{i-1 \text{ times}}, 1, 0, \ldots, 0), \quad i = 1, \ldots, n.$$

Since e is orthonormal, the dual basis d is equal to e.

In this situation, Corollary 3.7 gives

Corollary 4.1. *Let $v = (v_1, \ldots, v_n)$, $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n)$ and $z = (z_1, \ldots, z_n)$ be real sequences such that v and x are e-positive or e-negative. Assume $\langle x, v\rangle > 0$. Suppose that there exist index sets I_1 and I_2 with $I_1 \cup I_2 = I$, where $I = \{1, \ldots, n\}$, such that*

$$\frac{z_j}{v_j} \leqslant \frac{z_i}{v_i} \quad \textit{and} \quad \frac{y_j}{x_j} \leqslant \lambda \leqslant \frac{y_i}{x_i} \quad \textit{for } i \in I_1 \textit{ and } j \in I_2 \tag{29}$$

provided the denominators of the fractions are positive (or the reverse inequalities hold if the denominators are negative), where $\lambda = \sum_{k=1}^{n} y_k v_k / \sum_{k=1}^{n} x_k v_k$.

Then the following inequality holds

$$\sum_{k=1}^{n} z_k x_k \sum_{k=1}^{n} y_k v_k \leqslant \sum_{k=1}^{n} z_k y_k \sum_{k=1}^{n} x_k v_k. \tag{30}$$

According to Remark 3.8 the above result extends the applicability of a result due to Toader [10] from synchronous sequences to separable ones.

Example 4.2. For $v = x = (1, \ldots, 1)$, (30) leads to Chebyshev sum inequality (11) under the assumption that there exist I_1 and I_2 satisfying

$$z_j \leqslant z_i \quad \text{and} \quad y_j \leqslant \frac{1}{n}\sum_{k=1}^{n} y_k \leqslant y_i \quad \text{for } i \in I_1 \text{ and } j \in I_2 \tag{31}$$

(see (29)). If both z and y are assumed to be nondecreasing (or nonincreasing) then (31) is met, and no preliminary knowledge of I_1 and I_2 is required (cf. Example 3.3).

Letting $x = (1, \ldots, 1)$ and substituting $z_k v_k$ in place of z_k in (30), we get the *weighted version of Chebyshev inequality* (cf. [13, p. 317])

$$\sum_{k=1}^{n} z_k v_k \sum_{k=1}^{n} y_k v_k \leqslant \sum_{k=1}^{n} v_k \sum_{k=1}^{n} z_k y_k v_k \tag{32}$$

for any positive vector v, provided there exist I_1 and I_2 such that

$$z_j \leqslant z_i \quad \text{and} \quad y_j \leqslant \frac{\sum_{k=1}^n y_k v_k}{\sum_{k=1}^n v_k} \leqslant y_i \quad \text{for } i \in I_1 \text{ and } j \in I_2. \tag{33}$$

It is clear that (33) is met if z and y are nondecreasing (or nonincreasing).

Example 4.3. We now focus on the following result of Toader [13, Theorem 1]

$$\sum_{k=1}^n z_k v_k \sum_{k=1}^n y_k v_k \leqslant \frac{\left(\sum_{k=1}^n k v_k\right)^2}{\sum_{k=1}^n k^2 v_k} \sum_{k=1}^n z_k y_k v_k \tag{34}$$

for any positive vector v. This is a discrete version of *Andersson's inequality* (cf. [1, Theorem 2], [2, p. 1] and [5, Theorem 2.2]). We will derive it from (30) under the assumption that the left-hand side sums of (34) are nonnegative and that z and y are *star shaped*, that is the sequences $\{z_i/i\}_{i=1}^n$ and $\{y_i/i\}_{i=1}^n$ are nondecreasing [13, p. 318]. In this situation, the monotonicity of the sequences implies that the index sets are of the form $I_2 = \{1, \ldots, i_0\}$ and $I_1 = \{i_0+1, \ldots, n\}$ for some $0 \leqslant i_0 \leqslant n$ (with $I_1 = \emptyset$ for $i_0 = n$ and $I_2 = \emptyset$ for $i_0 = 0$).

To derive (34), we put $x_k = k$ into (29) and (30), and next we substitute $z_k v_k$ and $k v_k$ in place of z_k and v_k, respectively, obtaining

$$\sum_{k=1}^n k z_k v_k \sum_{k=1}^n k y_k v_k \leqslant \sum_{k=1}^n z_k y_k v_k \sum_{k=1}^n k^2 v_k. \tag{35}$$

Further, we apply (29) and (30) for $v_k = 1$ and substitute $k v_k$, $k^2 v_k$ and z_k/k in place of x_k, y_k and z_k, respectively, obtaining

$$\sum_{k=1}^n z_k v_k \sum_{k=1}^n k^2 v_k \leqslant \sum_{k=1}^n k z_k v_k \sum_{k=1}^n k v_k. \tag{36}$$

In a similar manner we get

$$\sum_{k=1}^n y_k v_k \sum_{k=1}^n k^2 v_k \leqslant \sum_{k=1}^n k y_k v_k \sum_{k=1}^n k v_k. \tag{37}$$

Multiplying (36) and (37), and using (35), we obtain (34).

We now give another interpretation of Corollary 3.7 for the nonorthogonal bases e and d defined in Example 3.6.

Corollary 4.4. *Let $v = (v_1, \ldots, v_n)$, $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n)$ and $z = (z_1, \ldots, z_n)$ be real sequences such that v is e-positive or e-negative and x is d-positive or d-negative. Assume $\langle x, v\rangle > 0$. Suppose that there exist index sets I_1 and I_2 with $I_1 \cup I_2 = I$, where $I = \{1, \ldots, n\}$, such that*

$$\frac{\sum_{k=1}^j z_k}{\sum_{k=1}^j v_k} \leqslant \frac{\sum_{k=1}^i z_k}{\sum_{k=1}^i v_k} \quad \textit{for } i \in I_1 \textit{ and } j \in I_2 \tag{38}$$

and

$$\frac{y_j - y_{j+1}}{x_j - x_{j+1}} \leqslant \lambda \leqslant \frac{y_i - y_{i+1}}{x_i - x_{i+1}} \quad \textit{for } i \in I_1 \textit{ and } j \in I_2 \tag{39}$$

provided the denominators of the fractions are positive (or the reverse inequalities hold if the denominators are negative), where $\lambda = \sum_{k=1}^n y_k v_k / \sum_{k=1}^n x_k v_k$ and $x_{n+1} = y_{n+1} = 0$.

Then inequality (30) *holds.*

Example 4.5. Let $x = (n, n-1, \ldots, 2, 1)$. Observe that x is d-positive. Assume that $\widetilde{y} = (y_1, \ldots, y_n, y_{n+1})$ is a *concave* sequence, that is $y_{i+1} \geqslant (y_i + y_{i+2})/2$ for $i = 1, \ldots, n-1$ (cf. [13, p. 318]). Let $y_{n+1} = 0$. Then $y = (y_1, \ldots, y_n)$ is concave and $2y_n \geqslant y_{n-1}$. It now follows that (39) is met for $I_2 = \{1, \ldots, i_0\}$ and $I_1 = \{i_0 + 1, \ldots, n\}$ for some $0 \leqslant i_0 \leqslant n$ depending on λ (with $I_1 = \emptyset$ for $i_0 = n$ and $I_2 = \emptyset$ for $i_0 = 0$).

If, in addition, the sequence $\left\{ \left(\sum_{k=1}^{i} z_k \right) / \sum_{k=1}^{i} v_k \right\}_{i=1}^{n}$ is nondecreasing with positive denominators (or nonincreasing with negative denominators), then (38) holds, and, in consequence, inequality (30) is true.

5. Further applications

We proceed to deduce, using our method, further results known in the literature. We focus on results of Toader [11,12] and of Wang and Luo [15] generalizing inequalities of Seitz and of Fujiwara.

An inequality of G. Seitz (see [15, pp. 2–3], cf. also [6,12]) asserts that

$$\frac{\sum_{i,j=1}^{n} a_{ij} x_i z_j}{\sum_{i,j=1}^{n} a_{ij} x_i v_j} \leqslant \frac{\sum_{i,j=1}^{n} a_{ij} y_i z_j}{\sum_{i,j=1}^{n} a_{ij} y_i v_j}, \tag{40}$$

where $A = (a_{ij})$ is an $n \times n$ real matrix such that

$$\begin{vmatrix} a_{ir} & a_{is} \\ a_{jr} & a_{js} \end{vmatrix} \geqslant 0 \quad \text{for } 1 \leqslant i < j \leqslant n \text{ and } 1 \leqslant r < s \leqslant n, \tag{41}$$

and $v = (v_1, \ldots, v_n)$, $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n)$ and $z = (z_1, \ldots, z_n)$ are real sequences satisfying the condition

$$\begin{vmatrix} y_i & y_j \\ x_i & x_j \end{vmatrix} \begin{vmatrix} z_r & z_s \\ v_r & v_s \end{vmatrix} \geqslant 0 \quad \text{for } 1 \leqslant i < j \leqslant n \text{ and } 1 \leqslant r < s \leqslant n. \tag{42}$$

Observe that (40) can be rewritten as the bifractional inequality

$$\frac{\langle xA, z \rangle}{\langle xA, v \rangle} \leqslant \frac{\langle yA, z \rangle}{\langle yA, v \rangle}, \tag{43}$$

where $\langle \cdot, \cdot \rangle$ is the standard inner product on $V = \mathbb{R}^n$ equipped with the standard orthonormal basis $e = d$. For simplicity, we assume the positivity of the denumerators.

So, to prove that (41)–(42) imply (40), it is sufficient to employ Corollary 3.7 and Remark 3.8 for vectors $\widetilde{x} = xA$ and $\widetilde{y} = yA$. For this end it is enough to verify (28) for pairs (z, v) and $(\widetilde{y}, \widetilde{x})$.

In fact, by the generalized Cauchy formulae and by (41) and (42), for $1 \leqslant r < s \leqslant n$, we have that

$$\begin{vmatrix} \widetilde{y}_r & \widetilde{y}_s \\ \widetilde{x}_r & \widetilde{x}_s \end{vmatrix} \begin{vmatrix} z_r & z_s \\ v_r & v_s \end{vmatrix} = \begin{vmatrix} (yA)_r & (yA)_s \\ (xA)_r & (xA)_s \end{vmatrix} \begin{vmatrix} z_r & z_s \\ v_r & v_s \end{vmatrix} = \sum_{1 \leqslant i < j \leqslant n} \begin{vmatrix} y_i & y_j \\ x_i & x_j \end{vmatrix} \begin{vmatrix} z_r & z_s \\ v_r & v_s \end{vmatrix} \begin{vmatrix} a_{ir} & a_{is} \\ a_{jr} & a_{js} \end{vmatrix} \geqslant 0,$$

which gives the desired synchronicity of the pairs. Now, inequalities (43) and (40) follow from Corollary 3.7 via Remark 3.8.

Moreover, assumptions (41)–(42) can be weakened as follows (see Corollary 4.1):

$$\frac{z_j}{v_j} \leqslant \frac{z_i}{v_i} \quad \text{and} \quad \frac{\widetilde{y}_j}{\widetilde{x}_j} \leqslant \lambda \leqslant \frac{\widetilde{y}_i}{\widetilde{x}_i} \quad \text{for } i \in I_1 \text{ and } j \in I_2, \tag{44}$$

for some index sets I_1 and I_2 such that $I_1 \cup I_2 = \{1, \ldots, n\}$, where $\lambda = \langle \widetilde{y}, v \rangle / \langle \widetilde{x}, v \rangle$ with $\langle \widetilde{x}, v \rangle > 0$, $v_l > 0$, $\widetilde{x}_l > 0$ and

$$\widetilde{x}_l = \sum_{k=1}^{n} x_k a_{kl} \quad \text{and} \quad \widetilde{y}_l = \sum_{k=1}^{n} y_k a_{kl} \quad \text{for } l = 1, \ldots, n.$$

Example 5.1. Let $V = \mathbb{R}^3$ and $x = v = (1, 1, 1)$, $y = (1, 2, 1)$ and $z = (1, 2, 3)$. Take

$$A = \begin{pmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{8} \\ \frac{1}{4} & \frac{1}{2} & \frac{3}{8} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{pmatrix}.$$

Then $\widetilde{x} = xA = (1, 1, 1)$, $\widetilde{y} = yA = (1\frac{1}{4}, 1\frac{1}{2}, 1\frac{3}{8})$ and $\lambda = 1\frac{3}{8}$.

It is easily seen that (42) is not valid and (41) does not hold, since

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = \frac{3}{16} \quad \text{and} \quad \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} = -\frac{1}{16}.$$

However, for index sets $I_1 = \{2, 3\}$ and $I_2 = \{1\}$ condition (44) is satisfied, and the Seitz inequality (40) is valid by Corollary 4.1. Thus (44) extends conditions (41)–(42).

In his paper [11] Toader generalized a Fujiwara's inequality. Namely, he proved that if the pairs (z, v) and (y, x) are synchronous then the following inequality holds

$$Q := \begin{vmatrix} \langle y, z\rangle_1 & \langle y, v\rangle_1 \\ \langle x, z\rangle_2 & \langle x, v\rangle_2 \end{vmatrix} + \begin{vmatrix} \langle y, z\rangle_2 & \langle y, v\rangle_2 \\ \langle x, z\rangle_1 & \langle x, v\rangle_1 \end{vmatrix} \geqslant 0, \tag{45}$$

where $\langle\cdot,\cdot\rangle_k$ is a form on a function algebra V induced by a positive (sub)linear functional $A_k : V \to \mathbb{R}$, $k = 1, 2$, i.e., $\langle u, w\rangle_k = A_k uw$ for $u, w \in V$. Observe that the case $A_1 = A_2$ gives $\langle\cdot,\cdot\rangle_1 = \langle\cdot,\cdot\rangle_2$ and leads to inequalities of type (1)–(2) (cf. also Corollary 3.7 and Remark 3.8).

Let $\langle\cdot,\cdot\rangle_k$, $k = 1, 2$, be two inner products on a linear space V with $\dim V = n < \infty$. One approach to prove (45) is to use a method employing dual bases in V. Assume

$$e^{(k)} = \{e_i^{(k)} : 1, \ldots, n\} \quad \text{and} \quad d^{(k)} = \{d_i^{(k)} : 1, \ldots, n\}, \quad k = 1, 2,$$

are two pairs of bases in V satisfying $\langle d_j^{(k)}, e_i^{(k)}\rangle_k = \delta_{ji}$ (the Kronecker delta), $i, j = 1, \ldots, n$. That is, $d^{(k)}$ is the dual basis of $e^{(k)}$ with respect to $\langle\cdot,\cdot\rangle_k$.

Then we can apply the identity (see (45))

$$Q = \sum_{1 \leqslant i,j \leqslant n} \begin{vmatrix} \langle e_i^{(1)}, z\rangle_1 & \langle e_j^{(2)}, z\rangle_2 \\ \langle e_i^{(1)}, v\rangle_1 & \langle e_j^{(2)}, v\rangle_2 \end{vmatrix} \begin{vmatrix} \langle d_i^{(1)}, y\rangle_1 & \langle d_j^{(2)}, y\rangle_2 \\ \langle d_i^{(1)}, x\rangle_1 & \langle d_j^{(2)}, x\rangle_2 \end{vmatrix}.$$

It is now readily clear that (45) holds, if the pairs (z, v) and (y, x) are *synchronous* (w.r.t. the inner products $\langle\cdot,\cdot\rangle_k$ and w.r.t. the pairs of dual bases d_k and e_k, $k = 1, 2$) in the sense that

$$\begin{vmatrix} \langle e_i^{(1)}, z\rangle_1 & \langle e_j^{(2)}, z\rangle_2 \\ \langle e_i^{(1)}, v\rangle_1 & \langle e_j^{(2)}, v\rangle_2 \end{vmatrix} \begin{vmatrix} \langle d_i^{(1)}, y\rangle_1 & \langle d_j^{(2)}, y\rangle_2 \\ \langle d_i^{(1)}, x\rangle_1 & \langle d_j^{(2)}, x\rangle_2 \end{vmatrix} \geqslant 0 \quad \text{for } 1 \leqslant i, j \leqslant n. \tag{46}$$

The result (45) can be updated to a "separable" version. Remind that $z \in V$ is v-separable on index sets $I_1^{(k)}$ and $I_2^{(k)}$ satisfying $I_1^{(k)} \cup I_2^{(k)} = \{1, \ldots, n\}$ (w.r.t. $e^{(k)}$ and $\langle\cdot,\cdot\rangle_k$), if there exists a scalar μ such that

$$\langle e_i^{(k)}, z - \mu v\rangle_k \geqslant 0 \quad \text{and} \quad \langle e_j^{(k)}, z - \mu v\rangle_k \leqslant 0 \quad \text{for } i \in I_1^{(k)} \text{ and } j \in I_2^{(k)} \tag{47}$$

(see (18)). Likewise, $y \in V$ is λ, x-separable on $I_1^{(k)}$ and $I_2^{(k)}$ (w.r.t. $d^{(k)}$ and $\langle\cdot,\cdot\rangle_k$), if

$$\langle d_i^{(k)}, y - \lambda x\rangle_k \geqslant 0 \quad \text{and} \quad \langle d_j^{(k)}, y - \lambda x\rangle_k \leqslant 0 \quad \text{for } i \in I_1^{(k)} \text{ and } j \in I_2^{(k)}. \tag{48}$$

If the denominators of the fractions below are positive then (47)–(48) can be equivalently rewritten, respectively, as (see (20)–(21))

$$\frac{\langle e_j^{(k)}, z\rangle_k}{\langle e_j^{(k)}, v\rangle_k} \leqslant \frac{\langle e_i^{(k)}, z\rangle_k}{\langle e_i^{(k)}, v\rangle_k} \quad \text{for } i \in I_1^{(k)} \text{ and } j \in I_2^{(k)}, \tag{49}$$

$$\frac{\langle d_j^{(k)}, y\rangle_k}{\langle d_j^{(k)}, x\rangle_k} \leqslant \lambda \leqslant \frac{\langle d_i^{(k)}, y\rangle_k}{\langle d_i^{(k)}, x\rangle_k} \quad \text{for } i \in I_1^{(k)} \text{ and } j \in I_2^{(k)}. \tag{50}$$

Corollary 5.2. *Under the above notation, let v and x be vectors in V such that $\langle x, v\rangle_k > 0$ for $k = 1, 2$. Assume there exist index sets $I_1^{(k)}$ and $I_2^{(k)}$ with $I_1^{(k)} \cup I_2^{(k)} = \{1, \dots, n\}$ such that* (47)–(48) *are satisfied for some $\mu \in \mathbb{R}$ and for $\lambda = \lambda_{3-k} := \langle y, v\rangle_{3-k}/\langle x, v\rangle_{3-k}$, $k = 1, 2$.*

Then inequality (45) *is valid for vectors v, x, y and z.*

Proof. We use the identity (see (45)):

$$Q = \sum_{k=1}^{2} \langle x, v\rangle_{3-k} \left(\sum_{i \in I_1^{(k)}} \langle e_i^{(k)}, z - \mu v\rangle_k \langle d_i^{(k)}, y - \lambda_{3-k} x\rangle_k + \sum_{j \in I_2^{(k)}} \langle e_j^{(k)}, z - \mu v\rangle_k \langle d_j^{(k)}, y - \lambda_{3-k} x\rangle_k \right).$$

It is now evident that the separability of the pairs (z, v) and (y, x) expressed by (47)–(48) gives (45). □

Example 5.3. Consider $V = \mathbb{R}^3$ with the forms

$$\langle a, b\rangle_1 = a_1b_1 + a_2b_2 + a_3b_3 \quad \text{and} \quad \langle a, b\rangle_2 = a_1b_1 + 2a_2b_2 + a_3b_3$$

for $a = (a_1, a_2, a_3)$, $b = (b_1, b_2, b_3) \in \mathbb{R}^3$.

Set $x = v = (1, 1, 1)$, $y = \left(1, 2, 1\frac{3}{4}\right)$ and $z = (1, 2, 3)$. Then

$$\lambda_1 = \frac{\langle y, v\rangle_1}{\langle x, v\rangle_1} = 1\frac{7}{12} \quad \text{and} \quad \lambda_2 = \frac{\langle y, v\rangle_2}{\langle x, v\rangle_2} = 1\frac{11}{16}.$$

Let $d^{(1)} = e^{(1)}$ be the standard orthonormal basis in $\mathbb{R}^3$ w.r.t. $\langle \cdot, \cdot\rangle_1$. In addition, we take $e^{(2)}$ and $d^{(2)}$ to be the bases defined by

$$e_1^{(2)} = (1, 0, 0), \quad e_2^{(2)} = (0, 1, 0), \quad e_3^{(2)} = (0, 0, 1),$$
$$d_1^{(2)} = (1, 0, 0), \quad d_2^{(2)} = (0, \tfrac{1}{2}, 0), \quad d_3^{(2)} = (0, 0, 1).$$

It is obvious that $e^{(2)}$ and $d^{(2)}$ are dual w.r.t. the form $\langle \cdot, \cdot\rangle_2$. Then we obtain

$$\frac{\langle e_1^{(1)}, z\rangle_1}{\langle e_1^{(1)}, v\rangle_1} = 1, \quad \frac{\langle e_2^{(1)}, z\rangle_1}{\langle e_2^{(1)}, v\rangle_1} = 2, \quad \frac{\langle e_3^{(1)}, z\rangle_1}{\langle e_3^{(1)}, v\rangle_1} = 3,$$
$$\frac{\langle d_1^{(1)}, y\rangle_1}{\langle d_1^{(1)}, x\rangle_1} = 1, \quad \frac{\langle d_2^{(1)}, y\rangle_1}{\langle d_2^{(1)}, x\rangle_1} = 2, \quad \frac{\langle d_3^{(1)}, y\rangle_1}{\langle d_3^{(1)}, x\rangle_1} = 1\frac{3}{4},$$

and

$$\frac{\langle e_1^{(2)}, z\rangle_2}{\langle e_1^{(2)}, v\rangle_2} = 1, \quad \frac{\langle e_2^{(2)}, z\rangle_2}{\langle e_2^{(2)}, v\rangle_2} = 2, \quad \frac{\langle e_3^{(2)}, z\rangle_2}{\langle e_3^{(2)}, v\rangle_2} = 3,$$
$$\frac{\langle d_1^{(2)}, y\rangle_2}{\langle d_1^{(2)}, x\rangle_2} = 1, \quad \frac{\langle d_2^{(2)}, y\rangle_2}{\langle d_2^{(2)}, x\rangle_2} = 2, \quad \frac{\langle d_3^{(2)}, y\rangle_2}{\langle d_3^{(2)}, x\rangle_2} = 1\frac{3}{4}.$$

Taking $I_1^{(1)} = \{2, 3\}$, $I_2^{(1)} = \{1\}$, $I_1^{(2)} = \{2, 3\}$ and $I_2^{(2)} = \{1\}$, we see that conditions (49)–(50) are fulfilled. Consequently, (47)–(48) hold and, by Corollary 5.2, (45) is met. On the other hand, (46) fails for $i = 2$ and $j = 3$.

In conclusion, condition (46) is not necessary but sufficient, and (47)–(48) are sufficient for inequality (45) to hold.

An interesting extension of (45) to four-determinantal version is that of Toader [12]. It would be nice to have a "separable" counterpart of it.

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